NONSTEADY HEAT TRANSFER IN SEMI-INFINITE REGION WITH NONLINEAR HEAT-ABSORPTION LAW

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The method of determining the temperature gradient at the boundary of a semi-infinite region proposed earlier for linear problems [1-3] is outlined as it applies to a nonlinear problem.

After an appropriate choice of scales of the variables, the problem with zero initial conditions describing the heat transfer in a semi-infinite region absorbing heat according to the law $aT + bT^2$ $(a,\ b > 0)$ may be written in the form

$$\frac{\partial T}{\partial t} - \frac{\partial^2 T}{\partial x^2} + k(T + T^2/2) = 0,\ 0 \leqslant x < \infty,\ 0 < t < \infty, \tag{1}$$

$$T|_{x=0} = T_s(t), \tag{2}$$

$$T|_{x=\infty} = 0, \tag{3}$$

$$T|_{t=0} = 0. \tag{4}$$

The absorption law is chosen in this form so as to allow an exponential representation for slight temperature deviations from the zero value: $T + T^2/2 \approx \exp T - 1$.

It is required to determine the temperature gradient at the boundary of the region, $q_s = (\partial T/\partial x)_{x=0}$. Earlier [1], the dependence between T and $q = \partial T/\partial x$ was found for an analogous problem associated with the linear equation

$$\frac{\partial T}{\partial t} - \frac{\partial^2 T}{\partial x^2} + \gamma(x,\ t)\,T = 0 \tag{5}$$

(where γ is an arbitrary function having all its derivatives with respect to the two arguments) in the form

$$-q = D^{1/2}T + \frac{\gamma}{2}D^{-1/2}T + \frac{\gamma'}{4}D^{-1}T + \left(\frac{\gamma''}{8} - \frac{\dot{\gamma}'}{8} - \frac{\gamma^2}{8}\right)D^{-3/2}T + \left(\frac{\gamma'''}{16} - \frac{\dot{\gamma}'}{8} - \frac{\gamma\gamma'}{4}\right)D^{-2}T + \ldots,$$

$$D^{\nu}T = \frac{1}{\Gamma(1-\nu)}\frac{d}{dt}\int_0^t T(x,\ \tau)(t-\tau)^{-\nu}\,d\tau,\ \nu < 1, \tag{6}$$

$$D^{\nu}t^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+1-\nu)}\,t^{\mu-\nu}.$$

Here, a prime denotes the derivative, with respect to the coordinate, and a dot the derivative with respect to time.

Setting $\gamma = k(1 + T/2)$ makes Eqs. (1) and (5) coincide. Substituting the same expression into Eq. (6), and assuming that T (by analogy with the linear problem) is an analytic function of x and has all its derivatives with respect to t, a relation between q and T for Eqs. (1)-(4) may be written. It is assumed that $T'' = \dot{T} + k(T + T^2/2)$ and also (by analogy with the linear problem) that the mixed derivatives $\partial^{m+n}T/\partial x^m \partial t^n$ do not depend on the order in which the differentiations are performed. The relation between q and T then takes the form

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$$
\begin{aligned}
-q = {} & D^{1/2}T + \frac{k}{2}\left(1 + \frac{T}{2}\right) D^{-1/2}T + \frac{k^2}{8}\, qD^{-1}T - \\
& -\left(\frac{k^2}{8} + \frac{k^2}{16}\, T\right) D^{-3/2}T - \left(\frac{k}{32}\,\dot{q} + \frac{3k^2}{32}\, q + \frac{k^2}{32}\, Tq\right) D^{-2}T - \\
& -\left(\frac{k^2}{64}\,\dot{T} + \frac{k^2}{64}\,\dot{T}T + \frac{3k^2}{128}\, q^2 - \frac{k^3}{16} - \frac{k^3}{64}\, T + \frac{3k^3}{128}\, T^2 + \right. \\
& \left. + \frac{k^3}{128}\, T^3\right) D^{-5/2}T + \left(\frac{k}{64}\,\ddot{q} + \frac{3k^2}{64}\,\dot{q} + \frac{k^2}{64}\, T\dot{q} - \right. \\
& \left. - \frac{k^2}{128}\,\dot{T}q + \frac{9k^3}{128}\, q - \frac{k^3}{64}\, T^2 q\right) D^{-3}T + \\
& + \left(\frac{k^2}{128}\,\ddot{T} + \frac{k^2}{128}\,\dot{T}^2 + \frac{k^2}{128}\, T\ddot{T} + \frac{7k^2}{256}\, q\dot{q} + \frac{5k^3}{128}\,\dot{T} + \right. \\
& + \frac{9k^3}{256}\, T\dot{T} + \frac{k^3}{128}\, T^2\dot{T} + \frac{3k^3}{128}\, q^2 - \frac{3k^3}{512}\, Tq^2 - \frac{5k^4}{128} + \\
& \left. + \frac{k^4}{256}\, T + \frac{3k^4}{128}\, T^2 + \frac{k^4}{512}\, T^3 - \frac{k^4}{512}\, T^4\right) D^{-7/2}T + \dots .
\end{aligned}
\tag{7}
$$

Equation (7), written for $x = 0$, gives the relation between a specified value $T_S(t)$ and the desired gradient at the boundary $q_S(t)$.

An explicit expression for q_S in terms of T_S will be sought in the form of a series in powers of k

$$
q_s = q_0 + kq_1 + k^2 q_2 + k^3 q_3 + \dots . \tag{8}
$$

Substituting Eq. (8) into Eq. (7) and equating expressions with the same power of k, recurrence relations for determining q_n are obtained

$$
\begin{aligned}
-q_0 = {} & D^{1/2}T_s, \\
-q_1 = {} & \left(\frac{1}{2} + \frac{T_s}{4}\right) D^{-1/2}T_s + \frac{q_0}{8}\, D^{-1}T_s - \frac{1}{32}\,\dot{q}_0 D^{-2}T_s + \frac{1}{64}\,\ddot{q}_0 D^{-3}T_s + \dots, \\
-q_2 = {} & \frac{1}{8}\, q_1 D^{-1}T_s - \left(\frac{1}{8} + \frac{1}{16}\, T_s\right) D^{-3/2}T_s - \\
& -\left(\frac{1}{32}\,\dot{q}_1 + \frac{3}{32}\, q_0 + \frac{1}{32}\, T_s q_0\right) D^{-2}T_s - \\
& -\left(\frac{1}{64}\,\dot{T}_s + \frac{1}{64}\,\dot{T}_s T_s + \frac{3}{128}\, q_0^2\right) D^{-5/2}T_s + \\
& + \left(\frac{1}{64}\,\ddot{q}_1 + \frac{3}{64}\,\dot{q}_0 + \frac{1}{64}\, T_s\dot{q}_0 - \frac{1}{128}\,\dot{T}_s q_0\right) D^{-3}T_s + \\
& + \left(\frac{1}{128}\,\ddot{T}_s + \frac{1}{128}\,\dot{T}_s^2 + \frac{1}{128}\, T_s\ddot{T}_s + \frac{7}{256}\, q_0\dot{q}_0\right) D^{-7/2}T_s + \dots, \\
-q_3 = {} & \frac{1}{8}\, q_2 D^{-1}T_s - \left(\frac{1}{32}\,\dot{q}_2 + \frac{3}{32}\, q_1 + \frac{1}{32}\, T_s q_1\right) D^{-2}T_s - \\
& -\left(\frac{3}{64}\, q_0 q_1 - \frac{1}{16} - \frac{1}{64}\, T_s + \frac{3}{128}\, T_s^3 + \frac{1}{128}\, T_s^3\right) D^{-5/2}T_s + \\
& + \left(\frac{1}{64}\,\ddot{q}_2 + \frac{3}{64}\,\dot{q}_1 + \frac{1}{64}\, T_s\dot{q}_1 - \frac{1}{128}\,\dot{T}_s q_1 + \frac{9}{128}\, q_0 - \right. \\
& \left. - \frac{1}{64}\, T_s^2 q_0\right) D^{-3}T + \left(\frac{7}{256}\, q_0\dot{q}_1 + \frac{7}{256}\, q_1\dot{q}_0 + \right. \\
& + \frac{5}{128}\,\dot{T}_s + \frac{9}{256}\, T_s\dot{T}_s + \frac{1}{128}\, T_s^2\dot{T}_s + \frac{3}{128}\, q_0^2 - \\
& \left. - \frac{3}{512}\, T_s q_0^2\right) D^{-7/2}T_s + \dots,
\end{aligned}
\tag{9}
$$

. . . .

Equations (8) and (9) give a solution of the formulated problem—an expression for the temperature gradient at the surface for specified change in surface temperature.

In principle, this solution may be obtained for Eqs. (1)-(4) by the method of successive approximation. However, in practice only q_0 and q_1 may be found. Determining subsequent approximations involves an extremely large volume of calculations, even for a specific function $T_S(t)$.

The method proposed here allows several terms of the expansion in Eq. (8) to be easily calculated, since the whole temperature field is not determined for each approximation.

Example 1

Consider the case of a stepwise change in temperature at the boundary: $T_S = \Theta = \text{const}$. It follows from Eq. (9) that

$$-q_0 = \Theta\pi^{-1/2}\,t^{-1/2},$$

$$-q_1 = \left[\Theta + \left(\frac{1}{2} - \frac{1}{8} - 0 - \frac{1}{128} - 0 - \frac{1}{512} - 0 - \ldots\right)\right]\pi^{-1/2}\,t^{1/2} \approx [\Theta + 0.365\,\Theta^2]\,\pi^{-1/2}\,t^{1/2}.$$

To check the proposed method, an accurate value of q_1 is found by the method of successive approximation, applied to Eqs. (1)-(4). It is found that

$$-q_1 = [\Theta + (1 - 2/\pi)\,\Theta^2]\,\pi^{-1/2}\,t^{1/2} \approx [\Theta + 0.363\,\Theta^2]\,\pi^{-1/2}\,t^{1/2}.$$

Comparison shows that the coefficient for Θ is determined accurately by the proposed method and that for Θ^2 with a relative error of 0.55%.

For the subsequent terms in the expansion in powers of k, according to Eq. (9), it is found that (to illustrate the rate of convergence of the series, each term of the expansion for D^{ν} is written separately, starting with the first significant term)

$$\begin{aligned}
-q_2 = {} & [0.167\,\Theta + (-0.125 - 0.083 + 0.055 - 0.004 + \\
& + 0.000 - \ldots)\,\Theta^2 + [-0.046 + 0.018 - 0.004 + \\
& + 0.001 - 0.001 + \ldots)\,\Theta^3]\,\pi^{-1/2}\,t^{3/2}, \\
-q_3 = {} & [0.033\,\Theta + (-0.021 + 0.000 + 0.070 + 0.008 - 0.016 + \\
& + 0.000 - \ldots)\,\Theta^2 + (-0.019 + 0.000 + 0.036 - 0.015 + \\
& + 0.000 + 0.001 - \ldots)\,\Theta^3 + (-0.004 + 0.000 + 0.006 - 0.002 + \\
& + 0.002 - 0.000 + \ldots)\,\Theta^4]\,\pi^{-1/2}\,t^{5/2},
\end{aligned}$$

. . . .

For example, for $\Theta = 1$

$$-q_s\pi^{1/2}t^{-1/2}\,(1 + 1.365kt - 0.348k^2t^2 + 0.079k^3t^3 - \ldots).$$

This is suitable for practical calculations with $kt < 1$.

The present method will now be applied formally to a problem of the type in Eqs. (1)-(4) involving an arbitrary nonlinear heat-conduction equation

$$\frac{\partial T}{\partial t} - \alpha\left(T, \frac{\partial T}{\partial x}\right)\frac{\partial^2 T}{\partial x^2} - \beta\left(T, \frac{\partial T}{\partial x}\right)\frac{\partial t}{\partial x} + \gamma\left(T, \frac{\partial T}{\partial x}\right)T = 0, \tag{10}$$

where α, β, and γ are analytic functions of the arguments, $\alpha \geq 0$.

Since it is impossible, at present, to give a rigorous justifications of the method as applied to nonlinear equations, the solution has been checked for several nonlinear problems whose accurate solution is known. The example which is simplest in computational terms will be given here.

Example 2

It may be confirmed by direct verification that the problem

$$\frac{\partial T}{\partial t}-\frac{\partial^2 T}{\partial x^2}+k\left(\frac{\partial T}{\partial x}\right)^2=0,\ 0\leqslant x<\infty,\ 0<t<\infty, \tag{11}$$

$$T|_{x=0}=1;\ T|_{x=\infty}=0;\ T|_{t=0}=0$$

has the accurate solution

$$T=-k^{-1}\ln\{1-[1-\exp(-k)]\operatorname{erfc}(2^{-1}xt^{-1/2})\},$$

$$-q_s=-\left.\frac{\partial T}{\partial x}\right|_{x=0}=k^{-1}(\exp k-1)\pi^{-1/2}t^{-1/2}=$$
$$=(1+k/2+k^2/6+k^3/24+\ldots)\pi^{-1/2}t^{-1/2}\approx$$
$$\approx(1+0.5000k+0.1667k^2+0.0416k^3+\ldots)\pi^{-1/2}t^{-1/2}.$$

The method proposed here will now be used to solve Eq. (11) setting $\alpha=1$, $\beta=-kT'$, $\gamma=0$ in Eq. (10). Using the formulas of [1], a relation analogous to Eq. (7) may be written, retaining terms up to a_6 (for $D^{-5/2}T$). The solution obtained is of the form

$$-q_s=(1+0.5000k+0.1691k^2+0.0421k^3+\ldots)\pi^{-1/2}t^{-1/2}.$$

Comparison with the accurate solution in Eq. (12) shows that the coefficient for k is calculated accurately and the coefficients for k^2 and k^3 with a relative error of 1.4 and 1.2%, respectively.

The conditions of applicability of the method are more constraining than for linear problems. From the method of constructing the solution itself, it follows that $T_S(t)$ must have derivatives of all orders in the interval $0<t<\infty$. (In linear problems, $T_S(t)$ may have a discontinuity of the first kind in the given range.) At the point $t=0$, $T_S(t)$ may have a discontinuity of the first kind even in nonlinear problems, since the δ-like properties arising on differentiation [see Eq. (7)] are compensated by higher-order zeros—the factors $D^{-\nu}T_S$.

Consideration has also been given to several examples in which α, β, and γ in Eq. (10) are infinitely differentiable, but not analytic functions of the arguments. It was found that the form of the solution in Eq. (9) is determined correctly, but the infinite series arising in calculating the constant factors have no observable tendency to converge.

Thus, the method here proposed is practically applicable when α, β, and γ are analytical functions of the arguments, and the surface temperature $T_S(t)$ is an infinitely differentiable function for all $t>0$. At the point $t=0$, a finite discontinuity in T_S is permitted.

NOTATION

D^{ν}, fractional-differentiation symbol; T, temperature; q, temperature gradient; k, parameter characterizing heat-transfer rate; x, t, coordinate and time; α, β, γ, coefficients of general heat-conduction equation; Θ, constant; Indices: s, surface.

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